S17ACF – NAG Fortran Library Routine Document

Note. Before using this routine, please read the Users' Note for your implementation to check the interpretation of bold italicised terms and other implementation-dependent details.

1 Purpose

S17ACF returns the value of the Bessel Function $Y_0(x)$, via the routine name.

2 Specification

```
real FUNCTION S17ACF(X, IFAIL)
INTEGER              IFAIL
real                 X
```

3 Description

This routine evaluates an approximation to the Bessel Function of the second kind $Y_0(x)$.

Note. $Y_0(x)$ is undefined for $x \leq 0$ and the routine will fail for such arguments.

The routine is based on four Chebyshev expansions:

For $0 < x \leq 8$,

$$Y_0(x) = \frac{2}{\pi} \ln x {\sum_{r=0}}' a_r T_r(t) + {\sum_{r=0}}' b_r T_r(t), \text{ with } t = 2\left(\frac{x}{8}\right)^2 - 1,$$

For $x > 8$,

$$Y_0(x) = \sqrt{\frac{2}{\pi x}} \left\{ P_0(x) \sin\left(x - \frac{\pi}{4}\right) + Q_0(x) \cos\left(x - \frac{\pi}{4}\right) \right\}$$

where $P_0(x) = {\sum_{r=0}}' c_r T_r(t)$,

and $Q_0(x) = \frac{8}{x} {\sum_{r=0}}' d_r T_r(t)$, with $t = 2\left(\frac{8}{x}\right)^2 - 1$.

For x near zero, $Y_0(x) \simeq \frac{2}{\pi}\left(\ln\left(\frac{x}{2}\right) + \gamma\right)$, where γ denotes Euler's constant. This approximation is used when x is sufficiently small for the result to be correct to ***machine precision***.

For very large x, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the routine fails. Such arguments contain insufficient information to determine the phase of oscillation of $Y_0(x)$; only the amplitude, $\sqrt{\frac{2}{\pi n}}$, can be determined and this is returned on soft failure. The range for which this occurs is roughly related to the ***machine precision***: the routine will fail if $x \gtrsim 1/$***machine precision*** (see the Users' Note for your implementation for details).

4 References

[1] Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* Dover Publications (3rd Edition)

[2] Clenshaw C W (1962) Mathematical tables *Chebyshev Series for Mathematical Functions* HMSO

5 Parameters

1: X — ***real*** *Input*

On entry: the argument x of the function.

Constraint: X > 0.0.

2: IFAIL — INTEGER *Input/Output*

On entry: IFAIL must be set to 0, −1 or 1. For users not familiar with this parameter (described in Chapter P01) the recommended value is 0.

On exit: IFAIL = 0 unless the routine detects an error (see Section 6).

6 Error Indicators and Warnings

Errors detected by the routine:

IFAIL = 1

X is too large. On soft failure the routine returns the amplitude of the Y_0 oscillation, $\sqrt{2/\pi x}$.

IFAIL = 2

X ≤ 0.0, Y_0 is undefined. On soft failure the routine returns zero.

7 Accuracy

Let δ be the relative error in the argument and E be the absolute error in the result. (Since $Y_0(x)$ oscillates about zero, absolute error and not relative error is significant, except for very small x.)

If δ is somewhat larger than the machine representation error (e.g., if δ is due to data errors etc.), then E and δ are approximately related by

$$E \simeq |xY_1(x)|\delta$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|xY_1(x)|$.

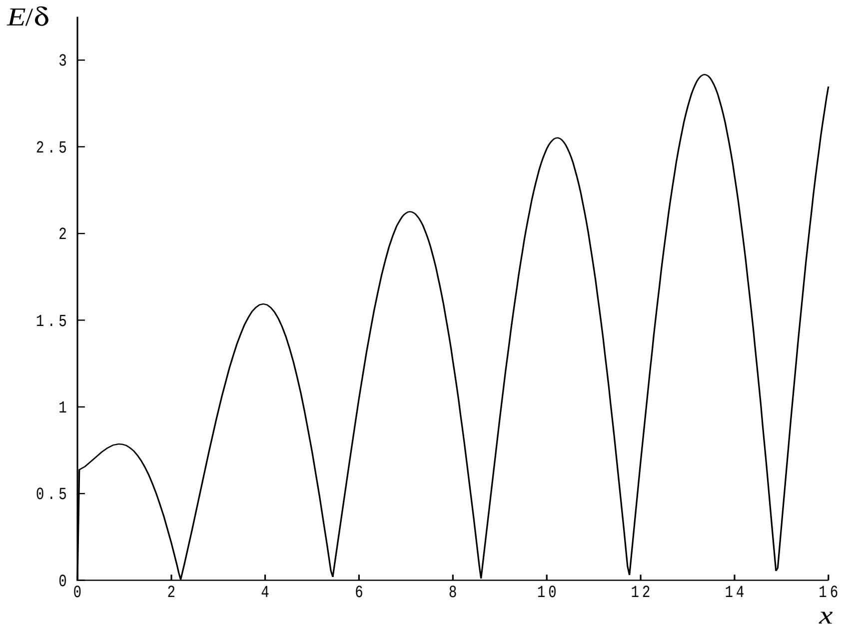

Figure 1

However, if δ is of the same order as the machine representation errors, then rounding errors could make E slightly larger than the above relation predicts.

For very small x, the errors are essentially independent of δ and the routine should provide relative accuracy bounded by the ***machine precision***.

For very large x, the above relation ceases to apply. In this region, $Y_0(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right)$. The amplitude $\sqrt{\frac{2}{\pi x}}$ can be calculated with reasonable accuracy for all x, but $\sin\left(x - \frac{\pi}{4}\right)$ cannot. If $x - \frac{\pi}{4}$ is written as $2N\pi + \theta$ where N is an integer and $0 \le \theta < 2\pi$, then $\sin\left(x - \frac{\pi}{4}\right)$ is determined by θ only. If $x \gtrsim \delta^{-1}$, θ cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of the inverse of ***machine precision***, it is impossible to calculate the phase of $Y_0(x)$ and the routine must fail.

8 Further Comments

None.

9 Example

The example program reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

9.1 Program Text

Note. The listing of the example program presented below uses bold italicised terms to denote precision-dependent details. Please read the Users' Note for your implementation to check the interpretation of these terms. As explained in the Essential Introduction to this manual, the results produced may not be identical for all implementations.

```
*     S17ACF Example Program Text
*     Mark 14 Revised.  NAG Copyright 1989.
*     .. Parameters ..
      INTEGER          NIN, NOUT
      PARAMETER        (NIN=5,NOUT=6)
*     .. Local Scalars ..
      real             X, Y
      INTEGER          IFAIL
*     .. External Functions ..
      real             S17ACF
      EXTERNAL         S17ACF
*     .. Executable Statements ..
      WRITE (NOUT,*) 'S17ACF Example Program Results'
*     Skip heading in data file
      READ (NIN,*)
      WRITE (NOUT,*)
      WRITE (NOUT,*) '     X           Y        IFAIL'
      WRITE (NOUT,*)
   20 READ (NIN,*,END=40) X
      IFAIL = 1
*
      Y = S17ACF(X,IFAIL)
*
      WRITE (NOUT,99999) X, Y, IFAIL
      GO TO 20
   40 STOP
*
99999 FORMAT (1X,1P,2e12.3,I7)
      END
```

9.2 Program Data

```
S17ACF Example Program Data
             0.0
             0.5
             1.0
             3.0
             6.0
             8.0
             10.0
             -1.0
             1000.0
```

9.3 Program Results

```
 S17ACF Example Program Results

      X           Y        IFAIL

    0.000E+00   0.000E+00      2
    5.000E-01  -4.445E-01      0
    1.000E+00   8.826E-02      0
    3.000E+00   3.769E-01      0
    6.000E+00  -2.882E-01      0
    8.000E+00   2.235E-01      0
    1.000E+01   5.567E-02      0
   -1.000E+00   0.000E+00      2
    1.000E+03   4.716E-03      0
```